Anderson Localizatio-Delocalization Transition

Integer Q Hall Effect & Topological Insulators

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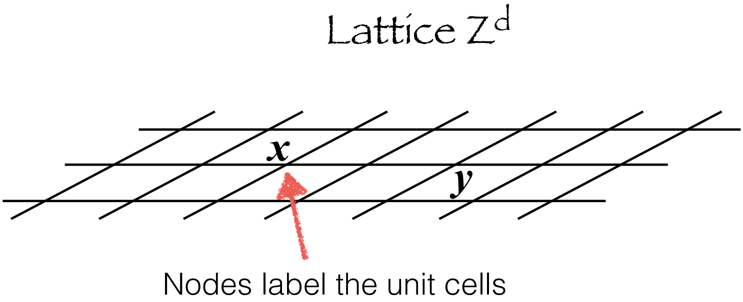
EMS-IAMP Summer School, July 11-15, 2016
Universality, Scaling Limits and Effective Theories

Work supported by U.S. NSF grant CAREER-DMR-1056168 and by Keck Foundation

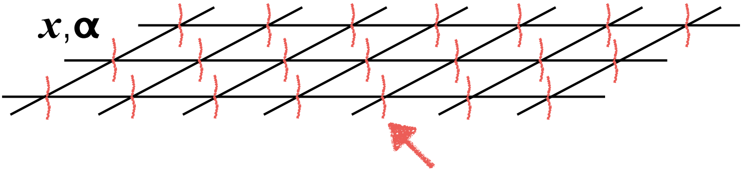
Part 1: Homogeneous Disordered Crystals

Starting point: The Discrete Lattice



The Fibers

Unit cells are usually complicated



Each lattice node carries a **fiber**

(fiber = finite dimensional Hilbert space)

The Underlying Atomic Potential

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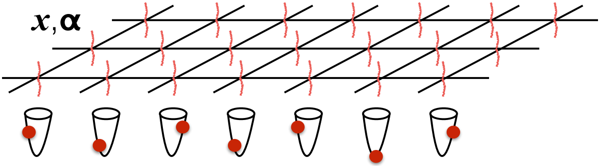
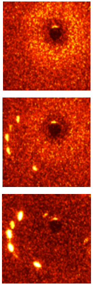
The atoms are never fixed



They giggle in an atomic force field



solidification of single-crystal

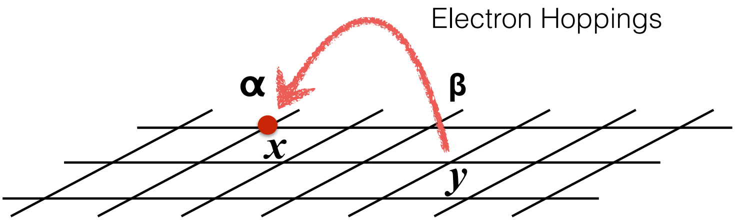
X-Ray diffraction pattern

$$\Omega = \left\{ \omega = \{\omega_x^\beta\}_{\beta, x \in \mathbb{Z}^d} \right\} = \prod_{x \in \mathbb{Z}^d} \Omega_0 \quad \text{(the configuration space)}$$

ω_x^β = displacement from equilibrium position of atom β in cell x.

$$\boxed{\text{Dynamical System: } (\Omega, \tau, \mathbb{P}), \quad \tau = (\tau_1, \ldots, \tau_d) = \text{action of } \mathbb{Z}^d \text{ on } \Omega}$$

The Hopping Processes



They generate all physical process

$$W_y \otimes |x\rangle\langle x| S_y, \quad W_y = \text{matrix acting on fibers}$$
$$S_y = \text{lattice shift by } y$$

Homogeneous Lattice Hamiltonians (with B-field)

The most general form:

$$H : \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d) \to \mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d)$$

$$H_\omega(B) = \sum_{y \in \mathcal{R}} \sum_{x \in \mathbb{Z}^d} W_y(\tau_x \omega) \otimes |x\rangle\langle x| U_y$$

$\mathcal{R} \subset \mathbb{Z}^d =$ finite hopping range; $U_y = e^{i\, y \wedge X} S_y =$ magnetic translations

Covariant property:

$$U_y H_\omega U_y^* = H_{\tau_y \omega}$$

w.r.t. a discrete ergodic dynamical system $(\Omega, \tau, \mathbb{Z}^d, \mathbb{P})$.

Remark:

The covariant property ensures that the macroscopic properties, including the macroscopic response functions, do not fluctuate from one disorder configuration to another.

Part 2: Classification of Condensed Matter Systems

A. P. Schnyder, S. Ryu, A. Furusaki, A. W. W. Ludwig, *Classification of topological insulators and superconductors in three spatial dimensions*, Phys. Rev. **B 78**, 195125 (2008).
A. Kitaev, *Periodic table for topological insulators and superconductors*, (Advances in Theoretical Physics: Landau Memorial Conference) AIP Conference Proceedings **1134**, 22-30 (2009).
S. Ryu, A. P. Schnyder, A. Furusaki, A. W. W. Ludwig, *Topological insulators and superconductors: tenfold way and dimensional hierarchy*, New J. Phys. **12**, 065010 (2010).

j	TRS	PHS	CHS	CAZ	0, 8	1	2	3	4	5	6	7
0	0	0	0	A	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
1	0	0	1	AIII		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
0	$+1$	0	0	AI	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$+1$	$+1$	1	BDI	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
2	0	$+1$	0	D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
3	-1	$+1$	1	DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
4	-1	0	0	AII	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
5	-1	-1	1	CII		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
6	0	-1	0	C			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
7	$+1$	-1	1	CI				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

- each $n \in \mathbb{Z}$ or $\mathbb{Z}_2$ defines a distinct macroscopic insulating phase: $\sigma_{xx} = 0$.
- the phases are separated by a bulk Anderson transition: $\sigma_{xx} > 0$
- $\sigma_{\parallel} > 0$ along any boundary cut into the crystals.

The Index Theorem for Bulk Projections (d = even)

Let d be even and let P_ω be a covariant projection such that:

$$\int_\Omega d\mathbb{P}(\omega)\ \Big\langle 0 \Big| \big|[X, P_\omega]\big|^d \Big| 0 \Big\rangle < \infty$$

Let $\Gamma_1, \ldots, \Gamma_2$ be irreducible rep of $\mathcal{C}l_d$. Then, $\mathbb{P}$-almost surely

$$F_\omega = P_\omega \left(\frac{X \cdot \Gamma}{|X|} \right)_{+-} P_\omega \in \text{Fredholm class}$$

and

$$\boxed{\mathrm{Ind}\, F_\omega = \Lambda_d \sum_{\rho \in S_d} (-1)^\rho \int_\Omega d\mathbb{P}_\omega \Big\langle 0 \Big| P_\omega \prod_{i=1}^{d} \imath \big[X_{\rho_i}, P_\omega\big] \Big| 0 \Big\rangle}$$

E. P., B. Leung, J. Bellissard, *The non-commutative n-th Chern number* ($n \geq 1$), J. Phys. A: Math. Theor. **46**, 485202 (2013).

Physical Interpretation

1. $P_F = \chi(H_\omega \leq \epsilon_F)$ be the Fermi projection.
2. $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$ be a set of indices, $|I| =$ even.
3. $$\mathrm{Ch}_I(P_F) = \Lambda_{|I|} \sum_{\rho \in S_I} (-1)^\rho \int_\Omega d\mathbb{P}_F \Big\langle 0 \Big| P_\omega \prod_{i=1}^{|I|} \imath [X_{\rho_i}, P_F] \Big| 0 \Big\rangle$$

Then

1. $\mathrm{Ch}_{\{i,j\}}(P_F) = \sigma_{ij}$ (Hall conductance for the (i,j)-plane)
2. $\mathrm{Ch}_{\{i,j\}\cup I}(P_F) = \partial_{\phi_{ij}} \mathrm{Ch}_I(P_F) \Rightarrow \mathrm{Ch}_I(P_F) = \partial_{\phi_{i_1 i_2}} \ldots \sigma_{i_{k-1} i_k}$

(non-linear Hall conductance)

E. P., H. Schulz-Baldes, *Bulk and Boundary Invariants for Complex Topological Insulators*, Springer monograph (2016).

The Index Theorem for Bulk Unitaries (d = odd)

Let d be odd and let U_ω be a covariant unitary such that:

$$\int_\Omega d\mathbb{P}(\omega)\ \Big\langle 0 \Big| \big|[X, U_\omega]\big|^d \Big| 0 \Big\rangle < \infty$$

Let E_+ be the spectral projection onto the positive spectrum of $X \cdot \Gamma$. Then, $\mathbb{P}$-almost surely

$$F_\omega = E_+ U_\omega E_+ \in \text{Fredholm class}$$

and

$$\boxed{\mathrm{Ind}\, F_\omega = \Lambda_d \sum_{\rho \in S_d} (-1)^\rho \int_\Omega d\mathbb{P}(\omega) \Big\langle 0 \Big| \prod_{i=1}^{d} \imath U_\omega^* \big[X_{\rho_i}, U_\omega\big] \Big| 0 \Big\rangle}$$

E. P., H. Schulz-Baldes, *Non-commutative odd Chern numbers and topological chiral phases*, J. Func. Analysis (in press).

Physical Interpretation

1. Chiral Symmetry: $\begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{pmatrix} H_\omega \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{pmatrix} = -H_\omega$
2. $$\mathrm{sgn}(H_\omega) = \begin{pmatrix} 0 & U_F^* \\ U_F & 0 \end{pmatrix}, \quad U_F = \text{Fermi unitary operator}$$
3. $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$ be a set of indices, $|I| = \text{odd}$.
4. $$\mathrm{Ch}_I(U_F) = \Lambda_{|I|} \sum_{\rho \in S_I} (-1)^\rho \int_\Omega d\mathbb{P}_\omega \Big\langle 0 \Big| \prod_{i=1}^{|I|} \imath U_F^* \big[X_{\rho_i}, U_F\big] \Big| 0 \Big\rangle$$

Then

1. $\mathrm{Ch}_{\{i\}}(U_F) = P_i^c$ (Vector of Macroscopic Chiral Polarization)
2. $\mathrm{Ch}_{\{i,j\} \cup I}(U_F) = \partial_{\phi_{ij}} \mathrm{Ch}_I(U_F) \Rightarrow \mathrm{Ch}_I(U_F) = \partial_{\phi_{i_1 i_2}} \ldots P_{i_k}^c$

E. P., H. Schulz-Baldes, *Bulk and Boundary Invariants for Complex Topological Insulators*, Springer monograph (2016).

Part 3: The algebraic framework of Jean Bellissard

The Algebra of Bulk Physical Observables $\mathcal{A}_d$

Definition (ϕ_{ij} = magnetic flux through the facets of unit cell)

The universal C*-algebra

$$\mathcal{A}_d \;=\; C^*\Big(C_N(\Omega), u_1, \cdots, u_d\Big)\,, \quad C_N(\Omega) = C(\Omega, M_{N\times N})$$

generated by the following commutation relations:

$$\left.\begin{array}{ll} u_i u_i^* = u_i^* u_i = \mathbf{1}, & i = 1, \ldots, d \\ u_i\, u_j \;= e^{i\phi_{ij}} u_j\, u_i, & i, j = 1, \ldots, d \end{array}\right\} non - commutative\ torus$$

$$f\, u_j \;=\; u_j\,(f \circ \tau_j)\,, \qquad \forall\ f \in C_N(\Omega)\,,\;\; j = 1, \ldots, d\,.$$

A generic element takes the form

$$a \;=\; \sum_{x \in \mathbb{Z}^d} a_x\, u_x, \quad a_x \in C_N(\Omega)\,, \quad u_x \;=\; u_1^{x_1} \cdots u_d^{x_d}\;.$$

Canonical Representation on $\mathbb{C}^N \otimes \ell^2(\mathbb{Z}^d)$

Proposition

$$\pi_\omega(u_j) \; = \; U_j, \quad j = 1, \ldots, d,$$

$$\pi_\omega(f) \; = \sum_{x \in \mathbb{Z}^d} f(\tau_x \omega) \otimes |x\rangle\langle x|, \quad \forall \, f \in C_N(\Omega),$$

defines a family $\{\pi_\omega\}_{\omega \in \Omega}$ of faithful representations.

For generic elements

$$\mathcal{A}_d \ni a = \sum_{y \in \mathbb{Z}^d} a_y \, u_y \longrightarrow \pi_\omega(a) = \sum_{x,y \in \mathbb{Z}^d} a_y(\tau_x \omega) \otimes |x\rangle\langle x| U_y.$$

All homogeneous lattice models can be generated from $\mathcal{A}_d$!

Fourier Calculus for Algebra of Observables

- Defined by the group of continuous $*$-automorphisms ρ_k induced by the $U(1)$ twists

$$u_j \to e^{\imath k_j} u_j, \qquad k_j \in [0, 2\pi],\ j = 1, \dots, d.$$

- The Fourier coefficients of $a \in \mathcal{A}_d$

$$\Phi_x(a) = \int_{\mathbb{T}^d} dk\ e^{-\imath \langle x | k \rangle} \rho_k(a u_x^*) \in C_N(\Omega), \quad x \in \mathbb{Z}^d.$$

- For a generic element $a \in \mathcal{A}_d$, the Cesàro sums converge to a

$$a^{(n)} = \sum_{x \in [-n, \dots, n]^d} \prod_{j=1}^{d} \left(1 - \frac{|x_j|}{n+1}\right) \Phi_x(a)\, u_x$$

Non-Commutative Calculus

Defined over $\mathcal{A}_d$ through the Fourier calculus:

Derivation:

$$\Phi_x(\partial_j a) = -\imath x_j \Phi_x(a), \;\; j = \overline{1, d}$$

For a generic element

$$a = \sum_{x \in \mathbb{Z}^d} a_x u_x \quad \rightarrow \quad \partial_j a = -\imath \sum_{x \in \mathbb{Z}^d} x_j \, a_x \, u_x$$

Integration:

$$\mathcal{T}(a) = \int_\Omega \mathbb{P}(d\omega) \, \mathrm{tr}\big\{\Phi_0(a)\big\}, \quad \mathcal{T}(a) = \int_\Omega d\mathbb{P}(\omega) \, \mathrm{tr}\{a_0(\omega)\}$$

The trace $\mathcal{T}$ over $\mathcal{A}_d$ is continuous, normalized and $\mathcal{T}(\partial_j a) = 0$.

- Chern Numbers (Bellissard et al, JMP 1994, EP et al 2013, EP et al 2014):

$$C_{even}(p) = \Lambda_d \sum_{\sigma \in S_d} (-1)^\sigma \mathcal{T}\big(p \prod_{i=1}^{d} \partial_{\sigma_i} p\big), \quad C_{odd}(u) = \Lambda_d \sum_{\sigma \in S_d} (-1)^\sigma \mathcal{T}\big(\prod_{i=1}^{d} u^* \partial_{\sigma_i} u\big)$$

- Finite-Temperature Kubo-formula (Schulz-Baldes & Bellissard in 1990's):

$$\sigma_{ij} = -\mathcal{T}\left((\partial_i h) * (\Gamma + \mathcal{L}_h)^{-1} \partial_j \Phi_{\mathrm{FD}}(h)\right).$$

- Electric polarization (Schulz-Baldes and Teufel in Comm. Math. Phys. 2012):

$$\Delta \mathbf{P} = \int_0^T dt \; \mathcal{T}\left(p(t)[\partial_t p(t), \nabla p(t)]\right)$$

- Orbital magnetization (Schulz-Baldes and Teufel in Comm. Math. Phys. 2012):

$$M_j = \frac{i}{2} \mathcal{T}\left(|h - \epsilon_F|[\partial_{j+1} p, \partial_{j+2} p]\right)$$

- Magneto-Electric Response in $d = 3$ (Leung and EP in J. Phys. A 2013):

$$\Delta\alpha = \tfrac{1}{2} \int dt \sum_{\sigma \in S_4} (-1)^\sigma \mathcal{T}\big(p \prod_{i=1}^{4} \partial_{\sigma_i} p\big), \quad \text{(4-th direction = time)}$$

My Goals for Today

Devise Explicit Numerical Algorithms for the Following Program:

Given a Hamiltonian $h \in \mathcal{A}_d$, compute correlation functions of the type:

$$\mathcal{T}\big(\partial^{\alpha_1} G_1(h)\, \partial^{\alpha_2} G_2(h) \ldots\big)$$

where G's are:

1. Smooth functions.
2. Piece-wise smooth, with singularities in Anderson localized spectrum.

Remark:

The principles have been announced in E.P., Appl. Math. Express (2013), where also the error estimates were derived explicitly for the disordered Hofstadter model.

Part 4: Key algebraic structures

The smooth sub-algebra

Let $C^n(\mathcal{A}_d)$ be the linear subspace spanned by those elements $a \in \mathcal{A}_d$ for which $\rho_\lambda(a)$ is n-times differentiable of λ. Then space of infinitely differentiable elements

$$\mathcal{A}_d^\infty \; = \; C^\infty(\mathcal{A}_d) \; = \; \bigcap_{n \geq 1} C^n(\mathcal{A}_d) \; ,$$

when endowed with the topology induced by the semi-norms:

$$\|a\|_\alpha \; = \; \|\partial^\alpha a\| \; , \qquad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \; , \;\; \alpha \; = \; (\alpha_1, \ldots \alpha_d) \; ,$$

becomes a dense Fréchet sub-algebra of $\mathcal{A}_d$, which is stable under the functional calculus with smooth functions.

Proposition

If $a \in \mathcal{A}_d^\infty$, then its Fourier coefficients have a fast decay property:

$$x^\alpha \|a_x\|_{C_N(\Omega)} \leq \|\partial^\alpha a\| \leq \infty, \quad x^\alpha = x^{\alpha_1} \ldots x^{\alpha_d}.$$

The Space of Periodic Disorder Configurations

$$\widetilde{\Omega} = \{\omega \in \Omega \mid \tau_j^{2L+1}\omega = \omega\}, \quad \tilde{\tau}_a = \tau_a|_{\widetilde{\Omega}}, \quad d\widetilde{\mathbb{P}}(\tilde{\omega}) = \prod_{x \in V_L} d\tilde{\omega}_x.$$

Proposition

Let $\tilde{\mathfrak{q}} : \Omega \to \widetilde{\Omega} \subset \Omega$ with $\tilde{\mathfrak{q}}\omega = \tilde{\omega}$ the unique $\tilde{\omega} \in \widetilde{\Omega}$ such that $\tilde{\omega}_x = \omega_x$ for all $x \in V_L$. Then:

1. The map $\tilde{\mathfrak{q}}$ is continuous, onto and idempotent, $\tilde{\mathfrak{q}}^2 = \tilde{\mathfrak{q}}$;
2. It induces the imbedding: $\mathfrak{i} : C_N(\widetilde{\Omega}) \hookrightarrow C_N(\Omega), \quad \mathfrak{i}(\tilde{f}) = \tilde{f} \circ \tilde{\mathfrak{q}}$
3. Let $\tilde{\mathfrak{p}}$ be the epimorphism of C^*-algebras:

   $$\tilde{\mathfrak{p}} : C_N(\Omega) \to C_N(\widetilde{\Omega}) \,, \quad \big(\tilde{\mathfrak{p}}(f)\big)(\tilde{\omega}) = f(\tilde{\omega}) \,.$$

   Then: $\tilde{\mathfrak{p}} \circ \mathfrak{i} = \mathrm{id}_{C_N(\widetilde{\Omega})} \,, \quad (\mathfrak{i} \circ \tilde{\mathfrak{p}})f = f \circ \tilde{\mathfrak{q}}.$
4. The following identity holds: $\int_\Omega d\mathbb{P}(\omega)\, \mathrm{tr}\big\{\big(\mathfrak{i}(\tilde{f})\big)(\omega)\big\} = \int_{\widetilde{\Omega}} d\widetilde{\mathbb{P}}(\tilde{\omega})\, \mathrm{tr}\{\tilde{f}(\tilde{\omega})\}.$

The Periodic Approximating Algebra $\tilde{\mathcal{A}}_d$

Definition:

$$\tilde{\mathcal{A}}_d \; = \; C^*\Big(C_N(\widetilde{\Omega}), u_1, \cdots, u_d\Big) \; ,$$

with same commutation relations as before.

Definition:

The non-commutative manifold $(\tilde{\mathcal{A}}_d, \tilde{\partial}, \widetilde{\mathcal{T}})$ is defined virtually the same way.

Proposition:

$$\tilde{\mathfrak{p}} : \mathcal{A}_d \to \tilde{\mathcal{A}}_d \; , \quad \sum_{x \in \mathbb{Z}^d} a_x u_x \to \sum_{x \in \mathbb{Z}^d} \tilde{a}_x u_x , \quad \tilde{a}_x = \tilde{\mathfrak{p}}(a_x) = a_x|_{\widetilde{\Omega}} \; .$$

is an epimorphism of C^*-algebras.

The Space of Finite-Volume Disorder Configurations

$$0 \longrightarrow \big((2L+1)\mathbb{Z}\big)^d \xrightarrow{\ i\ } \mathbb{Z}^d \underset{s}{\overset{\mathrm{ev}}{\rightleftarrows}} \widehat{\mathbb{Z}}^d = \big(\mathbb{Z}/(2L+1)\mathbb{Z}\big)^d \longrightarrow 0\,.$$

1. $\hat{x}$ denotes the class in $\widehat{\mathbb{Z}}^d$ of $x \in \mathbb{Z}^d$
2. The splitting map s, which is not unique, is fixed to $s(\hat{x}) = y$ where y is the unique point in V_L such that $\hat{y} = \hat{x}$.

Definition: The space of disorder configurations at finite volume

$$\widehat{\Omega} = \prod_{\hat{x}\in\widehat{\mathbb{Z}}^d} \Omega_0, \quad \hat{\tau}_y(\hat{\omega}) = \hat{\tau}_y\{\hat{\omega}_{\hat{x}}\}_{\hat{x}\in\widehat{\mathbb{Z}}^d} = \{\hat{\omega}_{\widehat{x-y}}\}_{\hat{x}\in\widehat{\mathbb{Z}}^d}, \quad d\widehat{\mathbb{P}}(\hat{\omega}) = \prod_{\hat{x}\in\widehat{\mathbb{Z}}^d} d\hat{\omega}_{\hat{x}}\,.$$

Proposition:

There exists an isomorphisms of dynamical systems

$$(\widetilde{\Omega}, \tilde{\tau}, \mathbb{Z}^d, d\widetilde{\mathbb{P}}) \simeq (\widehat{\Omega}, \hat{\tau}, \mathbb{Z}^d, d\widehat{\mathbb{P}}), \quad \tilde{\omega} = \{\tilde{\omega}_x\}_{x\in\mathbb{Z}^d} \to \hat{\mathfrak{q}}\tilde{\omega} \in \widehat{\Omega}, \quad (\hat{\mathfrak{q}}\tilde{\omega})_{\hat{x}} = \tilde{\omega}_{s(\hat{x})}.$$

The Finite Approximating Algebra

Definition:

$$\hat{\mathcal{A}}_d \;=\; C^*\Big(C_N(\widehat{\Omega}), \hat{u}_1, \cdots, \hat{u}_d\Big)\;,$$

with same commutation relations but the additional constraint:

$$(\tilde{u}_j)^{2L+1} = 1, \quad j = 1, \ldots, d.$$

The algebra is well defined only if $\phi_{ij} = \frac{2\pi}{2L+1} \times$ integer, since

$$\big(\hat{u}_i \hat{u}_j \hat{u}_i^*\big)^{2L+1} = e^{\imath(2L+1)\phi_{ij}}(\hat{u}_j)^{2L+1}.$$

Proposition:

$$\hat{\mathfrak{p}} : \tilde{\mathcal{A}}_d \to \hat{\mathcal{A}}_d\;, \quad \hat{\mathfrak{p}}(u_j) = \hat{u}_j\;, \quad \hat{\mathfrak{p}}(\tilde{f}) = \tilde{f} \circ \hat{\mathfrak{q}}^{-1}\;,$$

is a $*$-epimorphism of C^*-algebras.

Approximate Non-Commutative Calculus:

For generic $\hat{a} = \sum_{x \in V_L} \hat{a}_x \, \hat{u}_x$:

$$\hat{\partial}_j \hat{a} = -\imath \sum_{x \in V_L} x_j \hat{a}_x \, \hat{u}_x , \quad \widehat{\mathcal{T}}(\hat{a}) = \int_{\widehat{\Omega}} d\widehat{\mathbb{P}}(\hat{\omega}) \, \hat{a}_0(\hat{\omega}).$$

Under the canonical representations:

$$\hat{\pi}_{\hat{\omega}}(\hat{\partial}_j \hat{a}) = \sum_{\lambda^{2L+1}=1} c_\lambda \, \lambda^{\hat{X}} \hat{\pi}_{\hat{\omega}}(\hat{a}) \lambda^{-\hat{X}} \, , \quad c_\lambda = \begin{cases} \frac{\lambda^L}{1-\lambda}, & \lambda \neq 1, \\ 0, & \lambda = 1. \end{cases}$$

$$\widehat{\mathcal{T}}(\hat{a}) = \int_{\widehat{\Omega}} d\widehat{\mathbb{P}}(\hat{\omega}) \, \langle 0 | \hat{\pi}_{\hat{\omega}}(\hat{a}) | 0 \rangle = \frac{1}{|V_L|} \sum_{x \in V_L} \int_{\widehat{\Omega}} d\widehat{\mathbb{P}}(\tilde{\omega}) \, \langle x | \hat{\pi}_{\hat{\omega}}(\hat{a}) | x \rangle \, .$$

At this point we found the optimal replacement:

$$\imath [A_\omega, X_j] \to \sum_{\lambda^{2L+1}=1} c_\lambda \, \lambda^{\hat{X}} \hat{A}_{\hat{\omega}} \lambda^{-\hat{X}}, \quad A_\omega = \pi_\omega(a), \quad \hat{A}_{\hat{\omega}} = \hat{\pi}_{\hat{\omega}}\big(\hat{\mathfrak{p}} \circ \tilde{\mathfrak{p}}(a)\big)$$

This is the foundation for our finite-volume algorithm.

Fast Convergence to TD-Limit

Assumptions:

a1. The Hamiltonian h belongs to the smooth algebra $\mathcal{A}_d^\infty$.

a2. For any $K \in \mathbb{N}$, the Fourier coefficients of the Hamiltonian satisfy:

$$\|h_x(\omega) - h_x(\omega')\| \leq \frac{A_K}{(1+|V_M|)^K}\ , \quad 0 < A_K < \infty\ , \tag{1}$$

whenever $\omega_x = \omega'_x$ for $x \in V_M$, $M \in \mathbb{N}$.

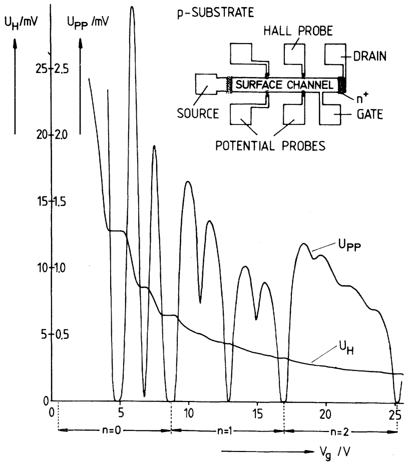
Theorem:

Let $h \in \mathcal{A}_d$ satisfying a1-a2. Define $\hat{h} \in \hat{A}_d$ as $\hat{h} = (\hat{\mathfrak{p}} \circ \tilde{\mathfrak{p}})(h)$. Then, for any $K \in \mathbb{N}$, $K \geq 2$, there exists the finite positive constant A_K such that:

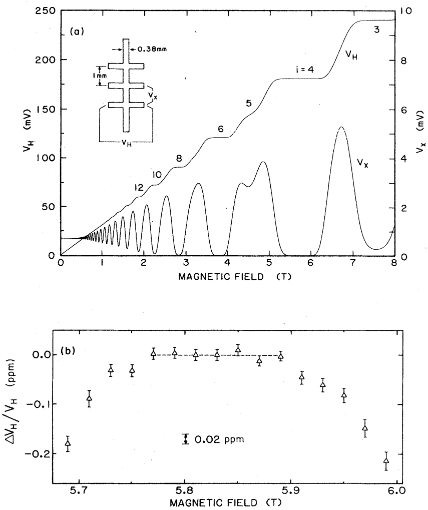
$$\left|\mathcal{T}\big(\partial^{\alpha_1} G_1(h) \ldots \partial^{\alpha_n} G_n(h)\big) - \widehat{\mathcal{T}}\big(\hat{\partial}^{\alpha_1} G_1(\hat{h}) \ldots \hat{\partial}^{\alpha_n} G_n(\hat{h})\big)\right| \leq \frac{A_K}{(1+|V_L|)^K}\ ,$$

where G_i's are smooth functions on the spectrum of h.

IQHE: A Crash Course

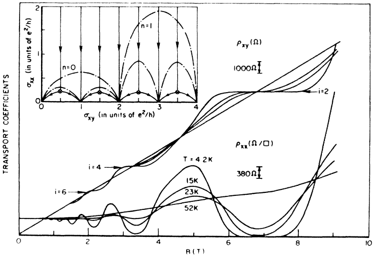


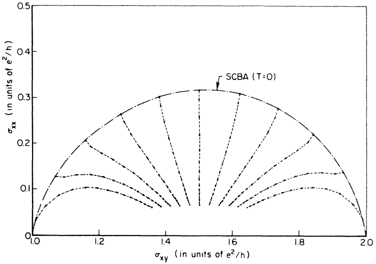
Klitzing et al (1980)



Cage et al. (1985)

IQHE: The Plateau-Plateau Transition





Critical Behavior
($\kappa = \frac{p}{2\nu}$, $\Gamma = 1/\tau_{\text{relaxation}} \sim T^p$):

$$\rho(E_F, T) = F\left((E_F - E_c)\left(\frac{T}{T_0}\right)^{-\kappa}\right)$$

$$\Lambda(E_F) \sim \frac{\alpha_0}{(E_F - E_c)^\nu}$$

Remark:

This scale-invariant behavior is observed only at T's which are low enough. When this happens, one says that the system entered the **quantum critical regime**.

The single-parameter scaling paradigm

Characteristics ($T = 0$):

▶ Diverging behavior of the localization length: $\xi \sim |E_F - E_F^c|^{-\nu}$.

▶ All physical quantities, measured on finite sample of size L, become functions of one parameter: ξ/L, near the transition.

In experiments:

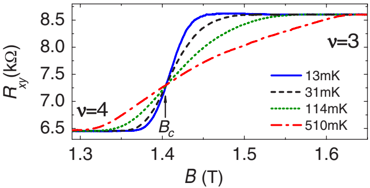
- one varies the temperature and L is practically ∞.
- L is replaced by the Thouless coherence length:

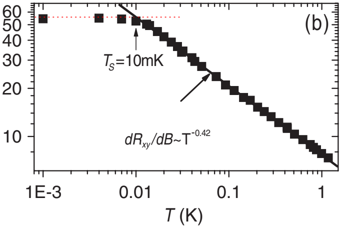
$$L_{\rm eff}(T) \sim \sqrt{\tau_{\rm relax}}, \quad \text{where the relaxation time } \tau_{\rm relax} \sim T^{-p}.$$

As a consequence, once T is low enough to enter the critical regime:

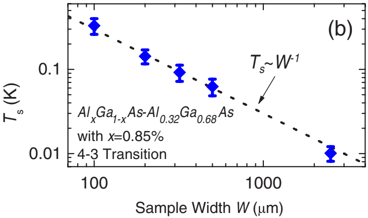
$$\boxed{\rho(E_F, T) = F\left((E_F - E_F^c)\left(\frac{T}{T_0}\right)^{-\kappa}\right), \quad \kappa = p/2\nu}$$

Experimental principles for evaluating κ and p (Tsui et al, PRL 2009)



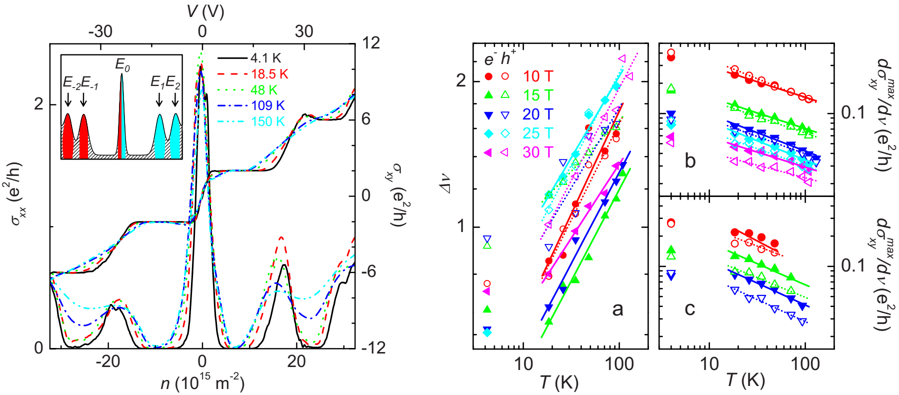


1. Saturation occurs at $L_{\text{eff}}(T) \approx L$.
2. By varying the sample-size, one can effectively map $L_{\text{eff}}(T) \sim T^{-p}$ and obtain p.
3. For PPT, one consistently obtain $p = 2$ and $\kappa = 0.42$ ($\rightarrow \nu = 2.38$).
4. New computations predicts $\nu = 2.6$!!!



Critical regime in graphene (Giesbers et al, PRB 2009)

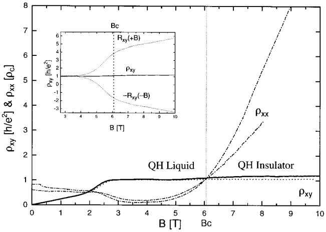
A look at the first and second (electron and hole) Landau bands:

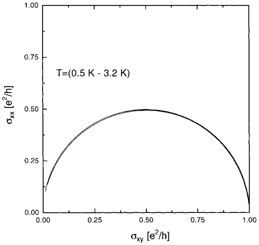


A value of $\kappa = 0.37$. Assuming $p = 2$, one obtains $\nu = 2.5$.

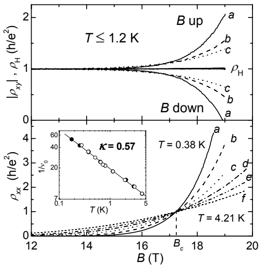
Critical Regime at the Plateau-Insulator Transition

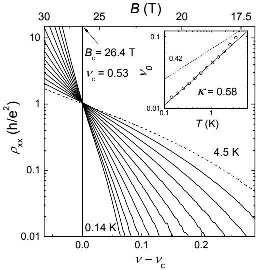
Tsui et al, Nature 1998





Pruisken et al, JP: Cond Mat 2006





Non-Commutative Kubo Formula

Due to Schulz-Baldes and Bellissard 1990')

$$\sigma_{ij}(\beta, E_F, \Gamma) = -\mathcal{T}\left(\partial_i h * (\Gamma + \mathcal{L}_h)^{-1}\partial_j \Phi_\beta(h - E_F)\right), \mathcal{L}_h[a] = i[h, a].$$

E_F	80×80	100×100	120×120	140×140	Exact
0.0	4.0339628247	4.0339630615	4.0339630708	4.0339630712	4.0339630712
-0.4	3.9394154619	3.9394154735	3.9394154621	3.9394154624	3.9394154624
-0.8	3.7040304262	3.7040301193	3.7040301310	3.7040301307	3.7040301307
-1.3	3.3684805414	3.3684801617	3.3684801517	3.3684801516	3.3684801516
-1.7	2.9522720814	2.9522713926	2.9522714007	2.9522714009	2.9522714009
-2.2	2.4678006935	2.4678005269	2.4678005093	2.4678005104	2.4678005104
-2.6	1.9239335953	1.9239338070	1.9239338090	1.9239338089	1.9239338089
-3.1	1.3274333126	1.3274333067	1.3274333084	1.3274333085	1.3274333085
-3.5	0.6854442914	0.6854442923	0.6854442923	0.6854442923	0.6854442923
-4.0	0.1086465150	0.1086465150	0.1086465150	0.1086465150	0.1086465150

σ_{11} at $kT = \Gamma = 0.1$, for a clean 2-dimensional lattice model.

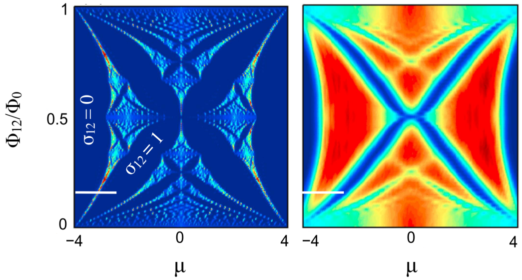
Disordered Hofstadter Model

Configuration space:

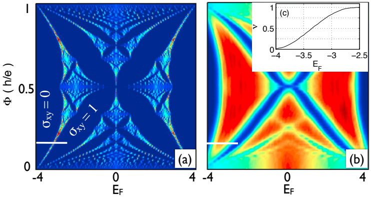
$$\Omega = \times_{x \in \mathbb{Z}^2} [-\tfrac{1}{2}, \tfrac{1}{2}], \quad \Omega \ni \omega = \{\omega_x\}_{x \in \mathbb{Z}^2}.$$

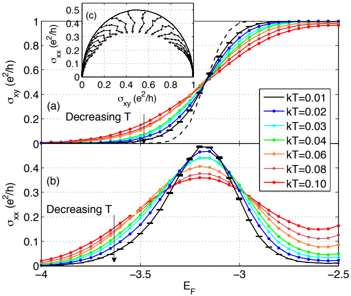
The Hamiltonian

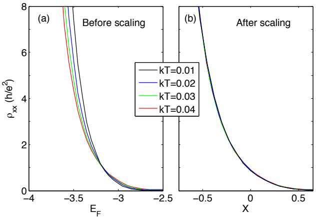
$$\mathcal{A}_d \ni h = u_1 + u_1^* + u_2 + u_2^* + \lambda V(\omega), \quad V(\omega) = \omega_0.$$

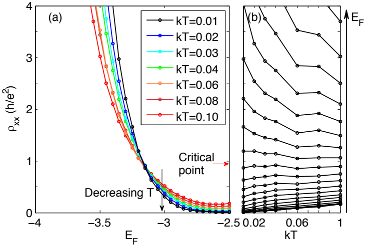


Plateau-Insulator transition in IQHE [Song and E.P., EPL (2014)]









Current-Current Correlation Function

Definition (Current-Current Correlation Measure):

$$\mathrm{e}^2 \int\limits_{\Delta E \times \Delta E'} \mathrm{d}m_{ij}(E,E') = \lim_{V\to\infty} \frac{1}{V} \sum_{(\epsilon_n,\epsilon_m)\in \Delta E\times\Delta E'} \langle \psi_n | J_i | \psi_m\rangle \langle \psi_m | J_j | \psi_n \rangle$$

or, for any continuous Φ, Φ'

$$\frac{\mathrm{e}^2}{\hbar^2} \int\limits_{\mathbb{R}\times\mathbb{R}} \Phi(E)\,\Phi'(E')\,\mathrm{d}m_{ij}(E,E') = \mathcal{T}\Big(\partial_i h * \Phi(h) * \partial_j h * \Phi'(h)\Big).$$

Definition (Current-Current Correlation Function):

$$dm_{ij}(E,E') = f_{ij}(E,E')dEdE', \quad f(E,E') = \frac{1}{d}\sum_i f_{ii}(E,E').$$

The CCC-function $f(E,E')$ is assumed to be continuous in both arguments.

Analysis on Current-Current Correlation Function

[E. P. & J. Bellissard, Ann. Phys. (2016)]

Direct Conductivity:

$$\sigma(\beta, E_F, \Gamma) = \frac{\mathrm{e}^2}{h} \int\limits_{\mathbb{R}\times\mathbb{R}} \frac{\Phi_\beta(E' - E_F) - \Phi_\beta(E - E_F)}{E - E'} \frac{4\pi\Gamma f(E, E')}{\Gamma^2 + (E - E')^2} \mathrm{d}E\mathrm{d}E'$$

Localization Length:

$$\Lambda^2(E_F) = \int_{-\infty}^{\infty} \frac{f(E_F, E)\mathrm{d}E}{(E_F - E)^2}$$

Asymptotic Behavior Near Critical-Point:

$$f(E, E') = g\left(\frac{E + E' - 2E_c}{(E - E')^{\kappa/p}}\right), \quad E, E' \sim E_c$$

Numerical Evaluation

Approximate Dirac-Delta function (analytic!):

$$\delta_\epsilon(t) = \frac{1/\epsilon}{\sqrt{2\pi}} e^{-(t/\epsilon)^2/2}$$

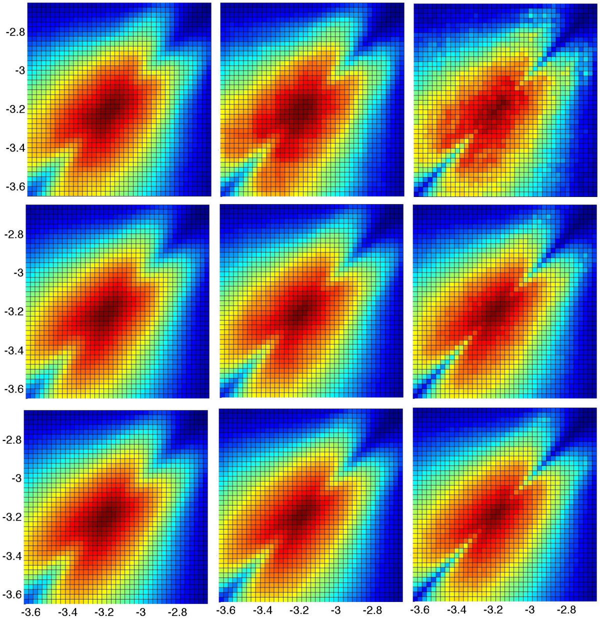
Smoothened CCC-function:

$$f_\epsilon(E, E') = \int_{\mathbb{R}\times\mathbb{R}} \delta_\epsilon(t - E)\delta_\epsilon(t - E')\,\mathrm{d}m(t, t')$$

$$f_\epsilon(E, E') = \frac{\hbar^2}{\mathrm{e}^2} \frac{1}{d} \sum_{i=1}^{d} \mathcal{T}\big(\partial_i h * \delta_\epsilon(h - E) * \partial_i h * \delta_\epsilon(h - E')\big)$$

Numerical approximate

$$\hat{f}_\epsilon(E, E') = \frac{\hbar^2}{\mathrm{e}^2} \frac{1}{d} \sum_{i=1}^{d} \widehat{\mathcal{T}}\big(\hat{\partial}_i \hat{h} * \delta_\epsilon(\hat{h} - E) * \hat{\partial}_i \hat{h} * \delta_\epsilon(\hat{h} - E')\big)$$



Rows correspond to:
$\mathcal{N} = 40$, 80 and 120.

Columns correspond to:

$\epsilon = 0.03$, 0.02 and 0.01.

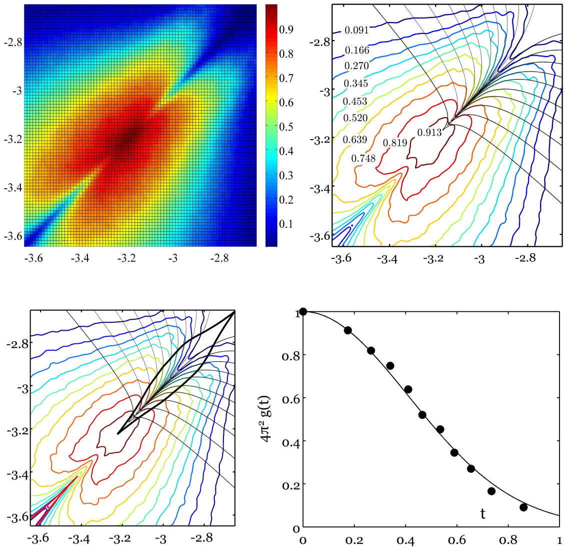
The predicted asymptotic behavior:

$(E, E' \sim E_c)$

$$f(E,E') = g\left(\frac{E+E'-2E_c}{(E-E')^{\kappa/p}}\right)$$

$$\downarrow$$

level sets: $\dfrac{E+E'-2E_c}{(E-E')^{\kappa/p}} = \text{const}$



Assumptions for $T = 0$ regime:

c1. The Hamiltonian $h \in \mathcal{A}_d$ and of the form:

$$w_y(\omega) = w_y + \sum_\alpha \omega_0^\alpha \lambda_y^\alpha \Rightarrow H_\omega = \sum_{x,y \in \mathbb{Z}^d} \left(w_y + \sum_\alpha \omega_x^\alpha \lambda_y^\alpha\right) \otimes |x\rangle\langle x| U_y .$$

c2. The Hamiltonian has mobility gaps Δ_i where the Aizenman-Molchanov bound holds:

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega) \, \|(h - z)_x(\omega)\|^s \leq A_s(\delta) e^{-\gamma_s(\delta)|x|} , \quad s \in (0,1), \ \delta > 0 , \qquad (2)$$

uniformly for all $z \in \mathbb{C} \setminus \sigma(h)$ with $\mathrm{dist}(z, \sigma(h) \setminus \Delta_i) \geq \delta$.

c3. The Aizenman-Molchanov bound holds uniformly in L for the finite-volumes approximations.

Fast Convergence to the TD Limit

Theorem:

Assume c1-c3 from above and define $\hat{h} \in \hat{\mathcal{A}}_d$ as $\hat{h} = (\hat{\mathfrak{p}} \circ \tilde{\mathfrak{p}})(h)$. Then, for any $K \in \mathbb{N}$, $K \geq 2$, there exists the finite positive constant A_K such that:

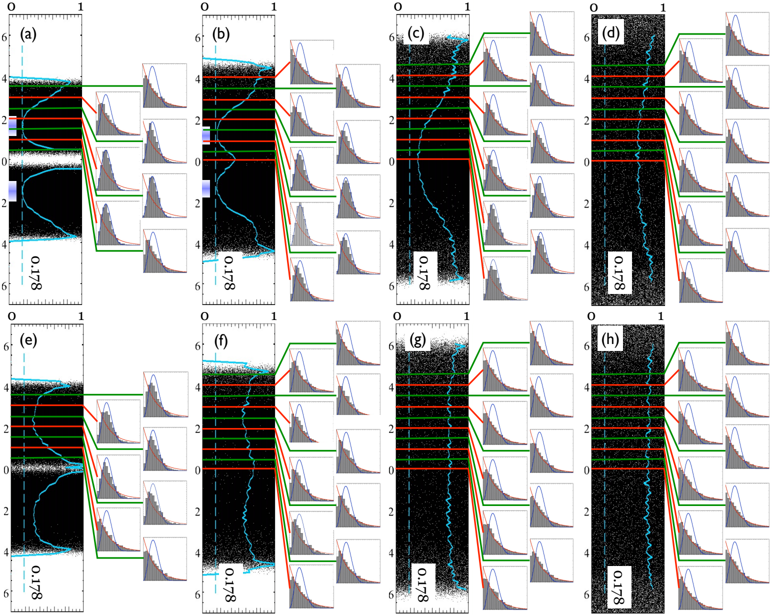
$$\Big| \mathcal{T}\big(\partial^{\alpha_1} G_1(h) \dots \partial^{\alpha_n} G_n(h)\big) - \widehat{\mathcal{T}}\big(\hat{\partial}^{\alpha_1} G_1(\hat{h}) \dots \hat{\partial}^{\alpha_n} G_n(\hat{h})\big) \Big| \leq \frac{A_K}{(1 + |V_L|)^K} ,$$

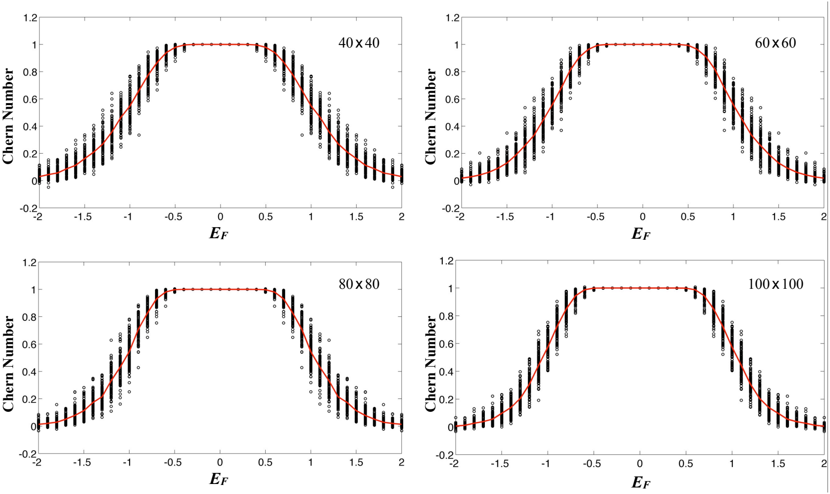
where G_i's are Borel functions that are smooth away from the mobility gaps of h.

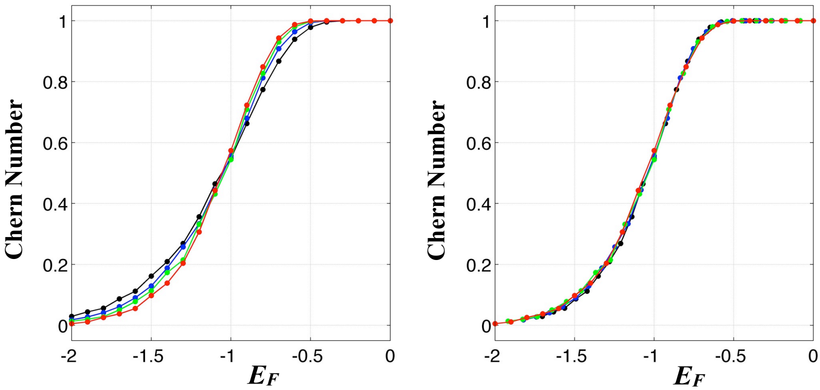
Disordered Haldane model ($\tau_n = \pm 1$)

$$H_\omega = \sum_{\langle n,m \rangle} |n\rangle\langle m| + 0.6\imath \sum_{\langle\langle n,m \rangle\rangle} \tau_n \big(|n\rangle\langle m| - |m\rangle\langle n| \big) + \lambda \sum_n \omega_n |n\rangle\langle n|.$$

on a honeycomb lattice (recall Alessandro's lecture).


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
0
1
0.178


40 x 40
60 x 60
80 x 80
100 x 100
Chern Number
E_F



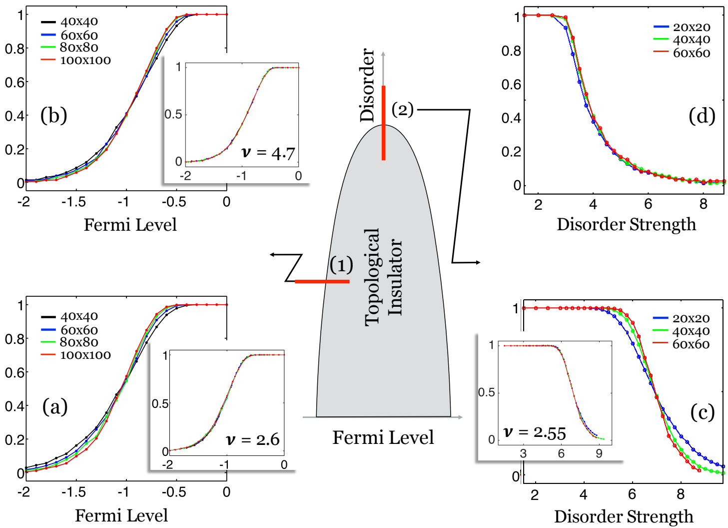
The Chern lines overlap almost perfectly after a rescaling of the energy axis

$$E \to E_c + (E - E_c) * (L/L_0)^\nu$$

($\nu = 2.6$, in line with the most recent estimates)

Table: Numerical values for average Chern numbers

Energy	40 × 40	60 × 60	80 × 80	100 × 100
-2.0000000000000000	0.0293885304649968	0.0183147848896676	0.0134785966919230	0.0055726403061233
-1.8999999999999999	0.0442301583027775	0.0274502505545331	0.0200229343621875	0.0112501012411246
-1.8000000000000000	0.0563736772645283	0.0416811880195335	0.0285382576963500	0.0259995275657507
-1.7000000000000000	0.0868202901241971	0.0612803850743208	0.0506852078002088	0.0377798251819264
-1.6000000000000001	0.1121154018269069	0.0905166860071905	0.0781754600177580	0.0554182299457663
-1.5000000000000000	0.1617580454580226	0.1291516191502659	0.1133966598848624	0.0977984662347778
-1.3999999999999999	0.2093536896403097	0.1883311262238442	0.1733092018533850	0.1386844139850113
-1.3000000000000000	0.2687556358733589	0.2575144956897765	0.2146703753513447	0.2040079233029510
-1.2000000000000000	0.3565352143319771	0.3333569482253110	0.3319133571108642	0.3066419928551302
-1.1000000000000001	0.4646789224167249	0.4444784219466996	0.4310440221933989	0.4427699238861748
-1.0000000000000000	0.5479958396159215	0.5561471440680733	0.5442615536532044	0.5738596277941682
-0.9000000000000000	0.6624275864985472	0.6798953821199148	0.7086514094234754	0.7228749266484203
-0.8000000000000000	0.7742005453064691	0.8124137607528051	0.8270271100278364	0.8487923693232788
-0.7000000000000000	0.8672349391630054	0.9079791895178040	0.9301639459675241	0.9432234611493278
-0.6000000000000000	0.9392873717233425	0.9636994770114942	0.9802652381114992	0.9872940741308633
-0.5000000000000000	0.9784417158133359	0.9935074963179980	0.9974987656403326	0.9988846769813913
-0.4000000000000000	0.9958865415757685	0.9992024708366942	0.9998527876642247	0.9999656328302596
-0.3000000000000000	0.9998184404341747	0.9999824660477071	0.9999988087144891	0.9999996457562911
-0.2000000000000000	0.9999952917010211	0.9999977443008894	0.9999999997655000	0.9999999999862120
-0.1000000000000000	0.9999999046002306	0.9999999998972079	0.9999999999998473	0.9999999999999849
0.0000000000000000	0.9999999963422543	0.9999999999988873	0.9999999999999996	0.9999999999999999


40x40
60x60
80x80
100x100
(b)
$\nu = 4.7$
Fermi Level
Disorder
(2)
20x20
40x40
60x60
(d)
Disorder Strength
(1)
Topological Insulator
Fermi Level
(a)
$\nu = 2.6$
Fermi Level
(c)
$\nu = 2.55$
Disorder Strength

AIII Model:

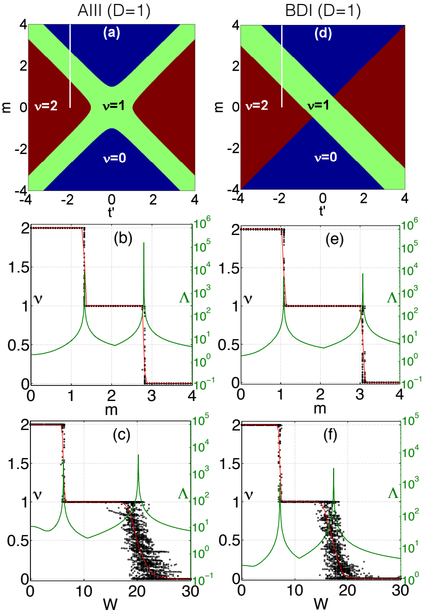
$$(H\psi)_x = m_x\ \hat{\sigma}_2\psi_x + \frac{1}{2}\ t_x[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+1} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-1}] + \frac{1}{2}\ t'[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+2} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-2}],$$

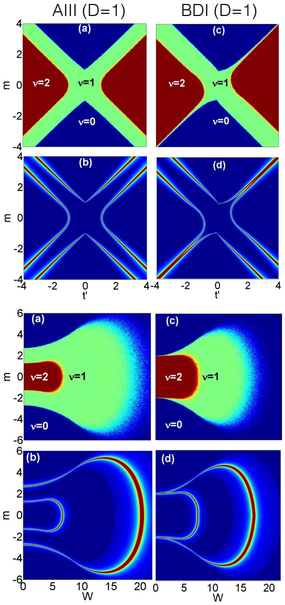
BDI Model:

$$(H\psi)_x = m_x\ \hat{\sigma}_1\psi_x + \frac{1}{2}\ t_x[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+1} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-1}] + \frac{1}{2}\ t'[(\hat{\sigma}_1 + i\hat{\sigma}_2)\psi_{x+2} + (\hat{\sigma}_1 - i\hat{\sigma}_2)\psi_{x-2}].$$

The disorder is present in the first-neighbor hopping and in the onsite potential:

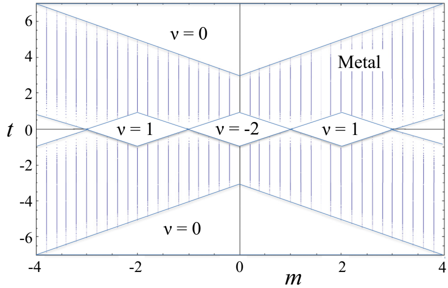
$$t_x = t + W_1\omega_x, \quad m_x = m + W_2\omega'_x, \quad \omega_x,\, \omega'_x \in [-\frac{1}{2}, \frac{1}{2}].$$


AIII (D=1)
BDI (D=1)
(a)
(b)
(c)
(d)
(e)
(f)
ν=2
ν=1
ν=0
m
t'
W
ν
Λ


AIII (D=1)
BDI (D=1)
(a)
(b)
(c)
(d)
ν=2
ν=1
ν=0
m
t'
W

The Model: (Γ's belong to $\mathcal{C}l_5$)

$$(H\psi)_x = \frac{1}{2}\sum_{j=1}^{3}\left\{\imath\Gamma_j(\psi_{x-e_j}-\psi_{x+e_j})+\Gamma_4(\psi_{x-e_j}+\psi_{x+e_j})\right\}$$
$$+\,\imath t\Gamma_1\Gamma_3\Gamma_4+(m+W\omega_x)\Gamma_4\psi_x$$



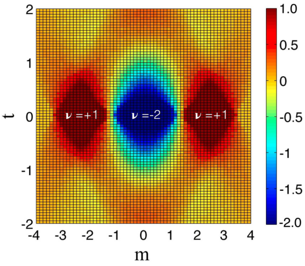


FIG. 3. (Color online) The phase diagrams in the phase space (m,t) at disorder strength $W = 4$. The computations were completed on a cubic lattice of $N = 16\times16\times16$ unit cells, following the procedure described in the text.

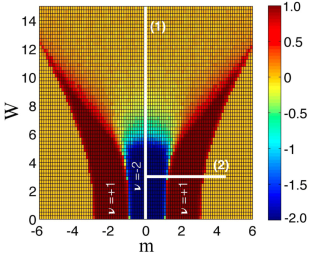


FIG. 4. (Color online) The phase diagrams in the phase space (m,W) at $t = 0$. The computations for ν were done with a cubic lattice of $N = 16\times16\times16$ unit cells.

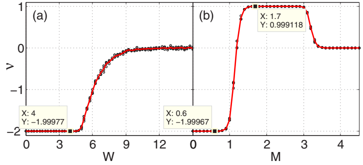


FIG. 5. (Color online) Evolution of the winding number ν with disorder W (a) and parameter m (b). The raw, unaveraged data for five disorder configurations are shown by the scattered points and the